Violation of Equalities in Bipartite Qutrits Systems

Hossein Movahhedian
*Department of Physics, Shahrood University of Technology, Seventh Tir Square, Shahrood, Iran**

We have recently shown that for the special case of a bipartite system with binary inputs and outputs there exist equalities in local theories which are violated by quantum theory. The amount of white noise tolerated by these equalities are twice that of inequalities. In this paper we will first introduce an inequality in bipartite qutrits systems which, if non-maximally entangled state is used instead of maximally entangled state, is violated more strongly by quantum theory. Hence reproducing the results obtained in the literature. We will then prove that our equalities in this case are violated by quantum theory too, and they tolerate much more white noise than inequalities.

October 16, 2008

I. INTRODUCTION

An interesting feature of local theories, discovered by John S. Bell in 1964 [1], is that for a local theory there exists an inequality which is violated by quantum theory. Bell inequalities have attracted more attention in recent years because it has been shown by Artur K. Ekert [2] that these inequalities can be used to establish a secure quantum key distribution. More interestingly, even if quantum theory is not correct, using the violations of Bell inequalities and the no-signaling principle it is possible to establish a secure quantum key distribution, see [3, 4]. Since the work of Bell many variations of Bell-type expressions have been introduced. These Bell-type expressions are in fact a linear combination of joint probabilities of outcomes in an experiment with two or more arms; so that on each arm two or more local variable settings are available and there are two or more outcomes for each variable setting. An inequality is then a Bell-type expression which is bounded by an upper and a lower bound. Among these are CHSH [5] and CH [6]. In this paper we restrict ourselves to experiments with two arms and two local variable settings on each arm and designate them by $l_1 l_2 \otimes r_1 r_2$ where $l_1(r_1)$ and $l_2(r_2)$ are the number of possible outcomes for the first and the second setting on the left(right) arm respectively. Two typical $22\otimes22$ inequalities are CHSH and CH inequalities in which the *amount of violation*, i.e. the difference between the value of a Bell expression according to quantum theory and its (extremum) value according to local theories, is 0.41421 and the *tolerance of white noise*, i.e. the maximum fraction of white noise admixture for which a Bell expression stops being violated, is 0.29298. For the case of $22\otimes22$ inequalities, Arthur Fine [7] proved that the necessary and sufficient condition for the existence of local realistic model is that the Bell/CH inequalities hold for the joint probabilities of the experiment.

In 1982, the experiment of Aspect et al. [8] showed that quantum theory is non-local. However, as experiments are not error-free, more efficient Bell-type expressions were introduced to test the non-locality of quantum theory more precisely, see [9] for details. The $33\otimes33$ case is the next version of Bell-type inequalities which has been studied widely in recent years, see [10–14]. The maximum value of the amount of violation and the tolerance of $33\otimes33$ inequalities are currently predicted to be 0.87293 and 0.30385 respectively for maximally entangled state [14].

Recently based on numerical calculations it has been shown in [17] that in $22\otimes22$ case, there exist Bell expressions in local theories with exact value, which we called them *equalities*, and has been proved that these equalities are violated more strongly than inequalities. In fact in our equalities the amount of violation and the tolerance of white noise admixture are 0.41421 and 0.58579 respectively. Here the tolerance is twice that of inequalities. However, as shown in [15, 16], for $33\otimes33$ case an inequality is more resistant to noise if non-maximally entangled state is used instead of maximally entangled state. Thus in this paper we consider the violation of our equalities using non-maximally entangled state. But prior to that we introduce an inequality in $33\otimes33$ case whose amount of violation and its tolerance of white noise admixture for non-maximally entangled state are a little more than that of maximally entangled state which is in agreement with the results of the above papers. Then we prove that there are equalities in $33\otimes33$ case whose tolerance of white noise admixture and violation factor would be much more than inequalities if non-maximally entangled state is used.

---

*Electronic address: `hossein_movahhedian@catholic.org`

II. BELL EXPRESSIONS AND THE EXPERIMENTAL SETUP

To verify the non-locality in quantum theory we consider a bipartite experiment in which one party, say Alice, performs two measurements $a \in \{1, 2\}$ with the outcomes $i_1 \in \{0, \cdots, l_1 - 1\}$ and $i_2 \in \{0, \cdots, l_2 - 1\}$ respectively. Similarly Bob performs two measurements $b \in \{1, 2\}$ with the outcomes $j_1 \in \{0, \cdots, r_1 - 1\}$ and $j_2 \in \{0, \cdots, r_2 - 1\}$ respectively. So the $l_1 \times l_2 \times r_1 \times r_2$ quantities $\gamma_{i_1 i_2 j_1 j_2}$ designate the double joint probabilities that measurements $a = 1$, $a = 2$, $b = 1$ and $b = 2$ give outcomes i_1, i_2, j_1, and j_2 respectively. Obviously

$$\sum_{i_1, i_2, j_1, j_2} \gamma_{i_1 i_2 j_1 j_2} = 1. \tag{1}$$

Now the joint probabilities, $P_{ab}^{i_a j_b}$, in terms of these double joint probabilities would be

$$P_{ab}^{i_a j_b} = \sum_{i_{a'}, j_{b'}} \gamma_{i_a i_{a'} j_b j_{b'}} \qquad \text{with } a', b' \in \{1, 2\},\ a' \neq a \text{ and } b' \neq b. \tag{2}$$

As has been proved in [17], two group of constraints, i.e. normalizability of joint probabilities and no-signaling reduces the number of independent P's to

$$N_I = (l_1 + l_2)(r_1 + r_2) - (l_1 + l_2 + r_1 + r_2 - 1). \tag{3}$$

Our numerical calculations show that N_I is 15 and 25 for $22 \otimes 33$ and $33 \otimes 33$ cases respectively, which once again confirms the above equation. Note that Eq. (3), as mentioned in [17], is not in agreement with the prediction in [18]. The reason is that one of the above constraints can be written in terms of the others and incidentally, this is not taken into account in [18]. We have fully discussed this in [17]. Also following the discussion in [17], in local theories a Bell expression can be written as

$$\mathbb{B} = \sum_{s,t,k,l} \lambda_{stkl} P_{st}^{kl} \tag{4}$$

where λ's are real (and usually integer) numbers. If we use Eq. (2) to write P_{st}^{kl} in terms of $\gamma_{i_1 i_2 j_1 j_2}$ we get

$$\mathbb{B} = \sum_{i_1, i_2, j_1, j_2} (\mu_{i_1 i_2 j_1 j_2} - \nu_{i_1 i_2 j_1 j_2}) \gamma_{i_1 i_2 j_1 j_2}, \quad \mu_{i_1 i_2 j_1 j_2} \neq \nu_{i_1 i_2 j_1 j_2} \tag{5}$$

where μ's and ν's are non-negative real numbers resulted after separating the positive and negative terms. Now the upper bound and the lower bound of $\mathbb{B}$ can be written as

$$-d \leq \mathbb{B} \leq c, \tag{6}$$

where $c(d)$ is the greatest of non-negative real numbers μ's(ν's).

The experiment we use is very close to that of Collins et al. introduced in [14], but instead of using a maximally entangled qutrit state as they did, we use the general bipartite state

$$|\Psi\rangle = \sum_{j,k=0}^{D-1} C_{jk} |j\rangle_A |k\rangle_B, \quad \sum_{j,k=0}^{D-1} |C_{jk}|^2 = 1. \tag{7}$$

Applying a phase transformation as below

$$|j\rangle_A \overset{Ph.T.}{\longrightarrow} e^{2\pi i (j\alpha/D)} |j\rangle_A, \tag{8}$$

$$|k\rangle_B \overset{Ph.T.}{\longrightarrow} e^{2\pi i (k\beta/D)} |k\rangle_B, \tag{9}$$

and the following discrete Fourier transformation

$$|j\rangle_A \overset{D.F.T.}{\longrightarrow} \frac{1}{\sqrt{D}} \sum_{m=0}^{D-1} e^{2\pi i (jm/D)} |m\rangle_A, \tag{10}$$

$$|k\rangle_B \overset{D.F.T.}{\longrightarrow} \frac{1}{\sqrt{D}} \sum_{n=0}^{D-1} e^{-2\pi i (kn/D)} |n\rangle_B, \tag{11}$$

would result the final state as

$$|\Psi\rangle = \frac{1}{D}\sum_{j,k,m,n=0}^{D-1} C_{jk} e^{(2\pi i/D)[(\alpha+m)j+(\beta-n)k]}|m\rangle_A|n\rangle_B. \tag{12}$$

where α and β are real constant numbers to be determined later. If Alice and Bob measure on $|m\rangle_A$ and $|n\rangle_B$ respectively, then the joint probabilities would be

$$P_{ab}^{mn} = \frac{1}{D^2}\left|\sum_{j,k=0}^{D-1} C_{jk} e^{(2i\pi/D)[(\alpha+m)j+(\beta-n)k]}\right|^2. \tag{13}$$

III. INEQUALITIES IN 33⊗33 SYSTEMS

For 33⊗33 systems we have listed all joint probabilities, P's, in terms of γ's in appendix A. Let's consider the following Bell expression in these systems

$$\begin{array}{rlllll} \mathbb{I} \; = & P_{11}^{00} & - \; P_{11}^{01} & - \; P_{11}^{10} & - \; 2P_{11}^{12} & - \; 2P_{11}^{20} \\ - & P_{11}^{21} & - \; P_{12}^{01} & - \; 2P_{12}^{02} & - \; P_{12}^{10} & + \; P_{12}^{11} \\ - & P_{12}^{21} & + \; P_{12}^{22} & - \; P_{21}^{00} & + \; P_{21}^{01} & - \; P_{21}^{11} \\ + & P_{21}^{12} & - \; P_{21}^{21} & - \; 2P_{21}^{22} & - \; 2P_{22}^{01} & - \; P_{22}^{02} \\ - & P_{22}^{10} & - \; 2P_{22}^{12} & - \; P_{22}^{20} & + \; P_{22}^{22}. & \end{array} \tag{14}$$

In appendix B we have shown that according to local theories we must have $-6 \le \mathbb{I} \le 0$.
If we define the matrix T_{mn} as

$$T_{mn} = \frac{1}{10}\begin{pmatrix} \sqrt{38} & 0 & 0 \\ 0 & \sqrt{24} & 0 \\ 0 & 0 & \sqrt{38} \end{pmatrix}, \tag{15}$$

and use the experiment discussed in section II, with $D = 3$, $C_{m-1,n-1} = T_{mn}$, $\alpha = \frac{1}{2}\delta_{a2}$ (δ_{xy} being Kronecker Delta), and $\beta = \frac{1}{4}\delta_{b1} - \frac{1}{4}\delta_{b2}$, the value of $\mathbb{I}$ predicted by quantum theory and its tolerance are 0.91485 and 0.31386 respectively. Hence reproducing the results in [15, 16]. These should be compared with the results obtained in [10–14] where, if maximally entangled states are used, the amount of violation and the tolerance for 33⊗33 (55⊗55) case are predicted to be 0.87293 (0.91054) and 0.30385 (0.31284) respectively.

IV. EQUALITIES IN 33⊗33 SYSTEMS

Equalities are built from Bell expressions which satisfy Eq. (6) with $d = 0$ and $c = 1$ which we will call *formal Bell expressions* from now on. Our numerical calculations show that formal Bell expressions in 32⊗22, 22⊗33, 32⊗32, and 33⊗33 cases, whether violated by quantum theory or not, are all built from 22⊗22 case by dividing one (or more) outcomes to two outcomes. As an example, the formal Bell expression for 32⊗22 case is obtained from 22⊗22 case simply by dividing one of the outcomes in setting labeled $a = 1$, on the left arm or Alice's measurements, to two outcomes. Similarly the formal Bell expression for 22⊗33 case is obtained from 22⊗22 case by dividing one of the outcomes in setting labeled $b = 1$ and one of the outcomes in setting labeled $b = 2$, on the right arm or Bob's measurements, each to two outcomes. Furthermore, if $\mathbb{E}$ is a formal Bell expression then its complement $\mathbb{E}_c$, whose sum with $\mathbb{E}$ add up to 1, can be obtained from $\mathbb{E}$ using one of the normalizability conditions, i.e. $\sum_{i,j} P_{ab}^{ij} = 1$. So based on these numerical results we conjecture that for higher dimensions the derivation of equalities must be similar. With this conjecture two formal Bell expressions in 33⊗33 case, which are complements of each other and are violated by quantum theory, can be built from 22⊗22 case easily. One such formal Bell expression is

$$\mathbb{E} = +P_{11}^{00} + P_{11}^{01} + P_{11}^{10} + P_{11}^{11} + P_{12}^{22} + P_{21}^{12} - P_{22}^{12}, \tag{16}$$

and its complement $\mathbb{E}_c$ is

$$\mathbb{E}_c = +P_{11}^{02} + P_{11}^{12} + P_{12}^{20} + P_{12}^{21} - P_{21}^{12} + P_{22}^{12}. \tag{17}$$

In appendix C we have shown directly that

$$|\mathbb{E}| + |\mathbb{E}_c| = 1. \tag{18}$$

It is interesting to note that if $\mathbb{E}$ and $\mathbb{E}_c$ are both positive, then, according to normalizability condition, both of them must be less than or equal to 1 and none of them would be violated by quantum theory. Now again using numerical calculations we have found that for the experiment discussed in section II, with

$$T_{mn} = \frac{1}{60}\begin{pmatrix} \sqrt{1302} & 0 & 0 \\ i\sqrt{60} & \sqrt{834} & 0 \\ i\sqrt{60} & i\sqrt{132} & \sqrt{1212} \end{pmatrix}, \tag{19}$$

$D = 3$, $C_{m-1,n-1} = T_{mn}$, $\alpha = \frac{7}{100}\delta_{a1} + \frac{62}{100}\delta_{a2}$ (δ_{xy} being Kronecker Delta), and $\beta = \frac{16}{100}\delta_{b1} - \frac{3}{10}\delta_{b2}$, the value of $\mathbb{E}_c$ predicted by quantum theory is -0.14895. So from Eq.s (16) and (18) we conclude that according to local theories we must have

$$\begin{aligned} |\mathbb{E}| &= |+P_{11}^{00} + P_{11}^{01} + P_{11}^{10} + P_{11}^{11} + P_{12}^{22} + P_{21}^{12} - P_{22}^{12}| \\ &= 0.85105. \end{aligned} \tag{20}$$

However, for these settings the value of $\mathbb{E}$ predicted by quantum theory is 1.14895.

Although equalities have not been tested in experiments yet (as far as the author is aware), they are testable because, as in the case of inequalites, in the testing of an equality the measurement of a Bell expression is relevant.

Now let's consider the tolerance of our equality, i.e. the maximum fraction of white noise admixture for which the Bell expression (16) stops being violated. The Werner state for three dimensional system is represented by

$$\rho = p\frac{\mathbb{1}}{9} + (1-p)|\Psi\rangle\langle\Psi|, \tag{21}$$

where Ψ is the entangled state as in Eq. (7) and p is the tolerance of the Bell expression. For such state the tolerance of our equality, i.e. Eq. (20), is 0.50203 which exceeds that of previous results by 0.19818 and with respect to inequality (14) the tolerance is increased by 0.18817. Note that in the presence of white noise the value of the Bell expression (17), denoted by $\mathbb{E}_c^{noisy}(p)$, would be

$$\mathbb{E}_c^{noisy}(p) = (m_c - n_c)\frac{p}{9} + (1-p)\mathbb{E}_c \tag{22}$$

where $\mathbb{E}_c$ is the value of the Bell expression according to quantum theory (i.e. for $\rho = |\Psi\rangle\langle\Psi|$) and for the above experimental setup it is -0.14895, and $m_c(n_c)$ is the number of joint probabilities, P_{ab}^{ij}, with positive (negative) sign, and from Eq. (17), $m_c - n_c = 4$. So for $p = 0.50203$ it is readily seen from Eq. (22) that $\mathbb{E}_c^{noisy}(p) = +0.14895$ and consequently according to local theories $\mathbb{E}^{noisy}(p)$ is still 0.85105.

The amount of violation of our equality, i.e. the difference between the value of the Bell expression (16) according to quantum theory and its value according to local theories, is $1.14895 - 0.85105 = 0.29790$ which is much less than that of inequality (14) and the one introduced in [14]. However, one should note that Eq. (14) contains 24 P's and its *range of violation*, i.e. the difference between the upper and the lower bound predicted by local realistic theories, denoted by R, is 6. But equality (20) only contains 7 P's and its range of violation is 0.85105, where we have used the exact value of the Bell expression for R which seems to be rational.

Therefore, we suggest the following generalized definition of violation factor, η, in terms of the amount of violation, δ, and the range of violation, R:

$$\eta = \frac{\delta + R}{R}. \tag{23}$$

We have used the range of violation in the above definition because there are inequalities with different number of P's but the same range of violation. With this definition the violation factor of our equality (20) is 1.35004 while that of inequality (14) and the one in [14] are 1.152475 and 1.14549 respectively.

V. CONCLUSION

In this paper we introduced an inequality in 33⊗33 case, i.e. in an experiment with two arms, two possible measurements on each arm and three possible outcomes for each measurement. We showed that if non-maximally

entangled states are used, the amount by which this inequality is violated and the amount of white noise admixture that can be added to a pure state so that it stops violating, i.e. its tolerance, is not only more than those in 33⊗33 case with maximally entangled state, but also more than those of 55⊗55 case, achieving the same results obtained by others.

Our numerical calculations shows that the total number of independent P's in 22⊗33 and 33⊗33 cases are 15 and 25 respectively which is in agreement with our analytical calculations in [17]. Note that the dimension of the space of P's predicted in [18] for the above cases are 14 and 24 respectively.

Based on our numerical calculations we conjectured how to derive the equalities in dimensions higher than two. Then we showed that equalities exist in 33⊗33 case and their tolerance of white noise and violation factor is much more than inequalities. This increasing of tolerance and violation factor in turn make the experiments and any other measurements related to non-locality much more easier.

However, we would like to emphasize that the tolerance of white noise and violation factor of equalities in 22⊗22 case are 1.52241 and 0.58579 respectively (see [17]), which are more than those of 33⊗33 equalities discussed in this paper (0.50203, 1.35004 respectively). So, in contrast to inequalities, according to our calculations the higher the dimension of the system, the lower the efficiency of the equalities. Note that for inequalities even for an infinite dimensional system the tolerance of white noise admixture and the violation factor currently predicted in the literature are 0.32656 and 1.16164 which are less than those of our 33⊗33 equalities.

VI. ACKNOWLEDGMENTS

We acknowledge the financial support of Shahrood University of Technology research Grant No. 24012.

[1] J. S. Bell (1964), *On the Einstein-Podolsky-Rosen paradox*, Physics 1, pp. 199-200.
[2] A. K. Ekert (1991), *Quantum cryptography based on Bell's theorem*, Phys. Rev. Lett. 67, 661-663.
[3] J. Barrett, L. Hardy, and A. Kent (2005), *No signalling and quantum key distribution*, Phys. Rev. Lett. 95, 010503.
[4] J. Barrett, A. Kent, and S. Pironio (2006), *Maximally non-local and monogamous quantum correlations*, Phys. Rev. Lett. 97, 170409.
[5] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt (1969), *Proposed experiment to test local hidden-variable theories*, Phys. Rev. Lett. 23, 880-884.
[6] J. F. Clauser, M. A. Horne (1974), *Experimental consequences of objective local theories*, Phys. Rev. D 10, 526-535.
[7] A. Fine (1982), *Hidden variables, joint probability, and the Bell inequalities*, Phys. Rev. Lett. 48, 291-295.
[8] A. Aspect, J. Dalibard, and G. Roger (1982), *Experimental test of Bell's inequalities using time -varying analyzers*, Phys. Rev. Lett. 49, 1804-1807.
[9] N. Gisin (2007), *Bell inequalities: many questions, a few answers*, quant-ph/0702021.
[10] D. Kaszlikowski, P. Gnacinski, M. Zukowski, W. Miklaszewski, A. Zeilinger (2000), *Violations of local realism by two entangled quNits are stronger than for two qubits*, Phys. Rev. Lett. 85, 4418-4421.
[11] J. L. Chen, D. Kaszlikowski, L. C. Kwek, M. Zukowski, and C. H. Oh (2001), *Entangled qutrits violate local realism stronger than qubits – analytical proof*, Phys. Rev. A 64, 052109.
[12] D. Kaszlikowski, L.C. Kwek, J.L. Chen, M. Zukowski, and C.H. Oh (2002), *Clauser-Horne inequality for qutrits*, Phys. Rev. A 65, 032118.
[13] T. Durt, D. Kaszlikowski, and M. Zukowski (2001), *Violations of local realism with quNits up to N=16*, Phys. Rev. A 64, 024101.
[14] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu (2002), *Bell inequalities for arbitrarily high dimensional systems*, Phys. Rev. Lett. 88, 040404.
[15] A. Acin, T. Durt, N. Gisin, and J.I. Latorre (2002), *Quantum non-locality in two three-level systems*, Phys. Rev. A 65, 052325.
[16] A. Acin, R. Gill, and N. Gisin (2005), *Optimal Bell tests do not require maximally entangled states*, Phys. Rev. Lett. 95, 210402.
[17] H. Movahhedian (2007), *Stronger violation of local theories with equalities*, J. Phys. A 40, 2839-2847.
[18] D. Collins and N. Gisin (2004), *A relevant two qubit Bell inequality to the CHSH inequality*, J. Phys. A 37, 1775-1787.

APPENDIX A: LIST OF JOINT PROBABILITIES

The joint probabilities, P_{ab}^{mn}, in terms of double joint probabilities, γ's in 33⊗33 case are:

$$
\begin{aligned}
P_{11}^{00} = {} & + \gamma_{0000} + \gamma_{0001} + \gamma_{0002} + \gamma_{0100} + \gamma_{0101} \\
& + \gamma_{0102} + \gamma_{0200} + \gamma_{0201} + \gamma_{0202} \\
P_{11}^{01} = {} & + \gamma_{0010} + \gamma_{0011} + \gamma_{0012} + \gamma_{0110} + \gamma_{0111} \\
& + \gamma_{0112} + \gamma_{0210} + \gamma_{0211} + \gamma_{0212} \\
P_{11}^{02} = {} & + \gamma_{0020} + \gamma_{0021} + \gamma_{0022} + \gamma_{0120} + \gamma_{0121} \\
& + \gamma_{0122} + \gamma_{0220} + \gamma_{0221} + \gamma_{0222} \\
P_{11}^{10} = {} & + \gamma_{1000} + \gamma_{1001} + \gamma_{1002} + \gamma_{1100} + \gamma_{1101} \\
& + \gamma_{1102} + \gamma_{1200} + \gamma_{1201} + \gamma_{1202} \\
P_{11}^{11} = {} & + \gamma_{1010} + \gamma_{1011} + \gamma_{1012} + \gamma_{1110} + \gamma_{1111} \\
& + \gamma_{1112} + \gamma_{1210} + \gamma_{1211} + \gamma_{1212} \\
P_{11}^{12} = {} & + \gamma_{1020} + \gamma_{1021} + \gamma_{1022} + \gamma_{1120} + \gamma_{1121} \\
& + \gamma_{1122} + \gamma_{1220} + \gamma_{1221} + \gamma_{1222} \\
P_{11}^{20} = {} & + \gamma_{2000} + \gamma_{2001} + \gamma_{2002} + \gamma_{2100} + \gamma_{2101} \\
& + \gamma_{2102} + \gamma_{2200} + \gamma_{2201} + \gamma_{2202} \\
P_{11}^{21} = {} & + \gamma_{2010} + \gamma_{2011} + \gamma_{2012} + \gamma_{2110} + \gamma_{2111} \\
& + \gamma_{2112} + \gamma_{2210} + \gamma_{2211} + \gamma_{2212} \\
P_{11}^{22} = {} & + \gamma_{2020} + \gamma_{2021} + \gamma_{2022} + \gamma_{2120} + \gamma_{2121} \\
& + \gamma_{2122} + \gamma_{2220} + \gamma_{2221} + \gamma_{2222} \\
P_{12}^{00} = {} & + \gamma_{0000} + \gamma_{0010} + \gamma_{0020} + \gamma_{0100} + \gamma_{0110} \\
& + \gamma_{0120} + \gamma_{0200} + \gamma_{0210} + \gamma_{0220} \\
P_{12}^{01} = {} & + \gamma_{0001} + \gamma_{0011} + \gamma_{0021} + \gamma_{0101} + \gamma_{0111} \\
& + \gamma_{0121} + \gamma_{0201} + \gamma_{0211} + \gamma_{0221} \\
P_{12}^{02} = {} & + \gamma_{0002} + \gamma_{0012} + \gamma_{0022} + \gamma_{0102} + \gamma_{0112} \\
& + \gamma_{0122} + \gamma_{0202} + \gamma_{0212} + \gamma_{0222} \\
P_{12}^{10} = {} & + \gamma_{1000} + \gamma_{1010} + \gamma_{1020} + \gamma_{1100} + \gamma_{1110} \\
& + \gamma_{1120} + \gamma_{1200} + \gamma_{1210} + \gamma_{1220} \\
P_{12}^{11} = {} & + \gamma_{1001} + \gamma_{1011} + \gamma_{1021} + \gamma_{1101} + \gamma_{1111} \\
& + \gamma_{1121} + \gamma_{1201} + \gamma_{1211} + \gamma_{1221} \\
P_{12}^{12} = {} & + \gamma_{1002} + \gamma_{1012} + \gamma_{1022} + \gamma_{1102} + \gamma_{1112} \\
& + \gamma_{1122} + \gamma_{1202} + \gamma_{1212} + \gamma_{1222} \\
P_{12}^{20} = {} & + \gamma_{2000} + \gamma_{2010} + \gamma_{2020} + \gamma_{2100} + \gamma_{2110} \\
& + \gamma_{2120} + \gamma_{2200} + \gamma_{2210} + \gamma_{2220} \\
P_{12}^{21} = {} & + \gamma_{2001} + \gamma_{2011} + \gamma_{2021} + \gamma_{2101} + \gamma_{2011} \\
& + \gamma_{2121} + \gamma_{2201} + \gamma_{2211} + \gamma_{2221} \\
P_{12}^{22} = {} & + \gamma_{2002} + \gamma_{2012} + \gamma_{2022} + \gamma_{2102} + \gamma_{2112} \\
& + \gamma_{2122} + \gamma_{2202} + \gamma_{2212} + \gamma_{2222} \\
P_{21}^{00} = {} & + \gamma_{0000} + \gamma_{0001} + \gamma_{0002} + \gamma_{1000} + \gamma_{1001} \\
& + \gamma_{1002} + \gamma_{2000} + \gamma_{2001} + \gamma_{2002} \\
P_{21}^{01} = {} & + \gamma_{0010} + \gamma_{0011} + \gamma_{0012} + \gamma_{1010} + \gamma_{1011} \\
& + \gamma_{1012} + \gamma_{2010} + \gamma_{2011} + \gamma_{2012} \\
P_{21}^{02} = {} & + \gamma_{0020} + \gamma_{0021} + \gamma_{0022} + \gamma_{1020} + \gamma_{1021} \\
& + \gamma_{1022} + \gamma_{2020} + \gamma_{2021} + \gamma_{2022} \\
P_{21}^{10} = {} & + \gamma_{0100} + \gamma_{0101} + \gamma_{0102} + \gamma_{1100} + \gamma_{1101} \\
& + \gamma_{1102} + \gamma_{2100} + \gamma_{2101} + \gamma_{2102} \\
P_{21}^{11} = {} & + \gamma_{0110} + \gamma_{0111} + \gamma_{0112} + \gamma_{1110} + \gamma_{1111} \\
& + \gamma_{1112} + \gamma_{2110} + \gamma_{2111} + \gamma_{2112} \\
P_{21}^{12} = {} & + \gamma_{0120} + \gamma_{0121} + \gamma_{0122} + \gamma_{1120} + \gamma_{1121} \\
& + \gamma_{1122} + \gamma_{2120} + \gamma_{2121} + \gamma_{2122} \\
P_{21}^{20} = {} & + \gamma_{0200} + \gamma_{0201} + \gamma_{0202} + \gamma_{1200} + \gamma_{1201} \\
& + \gamma_{1202} + \gamma_{2200} + \gamma_{2201} + \gamma_{2202}
\end{aligned}
$$

$$
\begin{aligned}
P_{21}^{21} = &+ \gamma_{0210} + \gamma_{0211} + \gamma_{0212} + \gamma_{1210} + \gamma_{1211}\\
&+ \gamma_{1212} + \gamma_{2210} + \gamma_{2211} + \gamma_{2212}\\
P_{21}^{22} = &+ \gamma_{0220} + \gamma_{0221} + \gamma_{0222} + \gamma_{1220} + \gamma_{1221}\\
&+ \gamma_{1222} + \gamma_{2220} + \gamma_{2221} + \gamma_{2222}\\
P_{22}^{00} = &+ \gamma_{0000} + \gamma_{0010} + \gamma_{0020} + \gamma_{1000} + \gamma_{1010}\\
&+ \gamma_{1020} + \gamma_{2000} + \gamma_{2010} + \gamma_{2020}\\
P_{22}^{01} = &+ \gamma_{0001} + \gamma_{0011} + \gamma_{0021} + \gamma_{1001} + \gamma_{1011}\\
&+ \gamma_{1021} + \gamma_{2001} + \gamma_{2011} + \gamma_{2021}\\
P_{22}^{02} = &+ \gamma_{0002} + \gamma_{0012} + \gamma_{0022} + \gamma_{1002} + \gamma_{1012}\\
&+ \gamma_{1022} + \gamma_{2002} + \gamma_{2012} + \gamma_{2022}\\
P_{22}^{10} = &+ \gamma_{0100} + \gamma_{0110} + \gamma_{0120} + \gamma_{1100} + \gamma_{1110}\\
&+ \gamma_{1120} + \gamma_{2100} + \gamma_{2110} + \gamma_{2120}\\
P_{22}^{11} = &+ \gamma_{0101} + \gamma_{0111} + \gamma_{0121} + \gamma_{1101} + \gamma_{1111}\\
&+ \gamma_{1121} + \gamma_{2101} + \gamma_{2111} + \gamma_{2121}\\
P_{22}^{12} = &+ \gamma_{0102} + \gamma_{0112} + \gamma_{0122} + \gamma_{1102} + \gamma_{1112}\\
&+ \gamma_{1122} + \gamma_{2102} + \gamma_{2112} + \gamma_{2122}\\
P_{22}^{20} = &+ \gamma_{0200} + \gamma_{0210} + \gamma_{0220} + \gamma_{1200} + \gamma_{1210}\\
&+ \gamma_{1220} + \gamma_{2200} + \gamma_{2210} + \gamma_{2220}\\
P_{22}^{21} = &+ \gamma_{0201} + \gamma_{0211} + \gamma_{0221} + \gamma_{1201} + \gamma_{1211}\\
&+ \gamma_{1221} + \gamma_{2201} + \gamma_{2211} + \gamma_{2221}\\
P_{22}^{22} = &+ \gamma_{0202} + \gamma_{0212} + \gamma_{0222} + \gamma_{1202} + \gamma_{1212}\\
&+ \gamma_{1222} + \gamma_{2202} + \gamma_{2212} + \gamma_{2222}.
\end{aligned}
$$

APPENDIX B: THE RANGE OF VIOLATION OF INEQUALITY $\mathbb{I}$

The range of violation of inequality (14), i.e. $\mathbb{I}$, can be found easily if we write $\mathbb{I}$ in the form of Eq. (5). We have

$$
\begin{aligned}
\mathbb{I} \;=\; &+ P_{11}^{00} - P_{11}^{01} - P_{11}^{10} - 2P_{11}^{12} - 2P_{11}^{20}\\
&- P_{11}^{21} - P_{12}^{01} - 2P_{12}^{02} - P_{12}^{10} + P_{12}^{11}\\
&- P_{12}^{21} + P_{12}^{22} - P_{21}^{00} + P_{21}^{01} - P_{21}^{11}\\
&+ P_{21}^{12} - P_{21}^{21} - 2P_{21}^{22} - 2P_{22}^{01} - P_{22}^{02}\\
&- P_{22}^{10} - 2P_{22}^{12} - P_{22}^{20} + P_{22}^{22}.
\end{aligned}
$$

Using P's as defined in appendix A, the inequality $\mathbb{I}$ can be written in terms of γ's as below

$$
\begin{aligned}
\mathbb{I} = &+ \gamma_{0000} + \gamma_{0001} + \gamma_{0002} + \gamma_{0100} + \gamma_{0101}\\
&+ \gamma_{0102} + \gamma_{0200} + \gamma_{0201} + \gamma_{0202}\\
&- \gamma_{0010} - \gamma_{0011} - \gamma_{0012} - \gamma_{0110} - \gamma_{0111}\\
&- \gamma_{0112} - \gamma_{0210} - \gamma_{0211} - \gamma_{0212}\\
&- \gamma_{1000} - \gamma_{1001} - \gamma_{1002} - \gamma_{1100} - \gamma_{1101}\\
&- \gamma_{1102} - \gamma_{1200} - \gamma_{1201} - \gamma_{1202}\\
&- 2\gamma_{1020} - 2\gamma_{1021} - 2\gamma_{1022} - 2\gamma_{1120} - 2\gamma_{1121}\\
&- 2\gamma_{1122} - 2\gamma_{1220} - 2\gamma_{1221} - 2\gamma_{1222}\\
&- 2\gamma_{2000} - 2\gamma_{2001} - 2\gamma_{2002} - 2\gamma_{2100} - 2\gamma_{2101}\\
&- 2\gamma_{2102} - 2\gamma_{2200} - 2\gamma_{2201} - 2\gamma_{2202}\\
&- \gamma_{2010} - \gamma_{2011} - \gamma_{2012} - \gamma_{2110} - \gamma_{2111}\\
&- \gamma_{2112} - \gamma_{2210} - \gamma_{2211} - \gamma_{2212}\\
&- \gamma_{0001} - \gamma_{0011} - \gamma_{0021} - \gamma_{0101} - \gamma_{0111}\\
&- \gamma_{0121} - \gamma_{0201} - \gamma_{0211} - \gamma_{0221}\\
&- 2\gamma_{0002} - 2\gamma_{0012} - 2\gamma_{0022} - 2\gamma_{0102} - 2\gamma_{0112}\\
&- 2\gamma_{0122} - 2\gamma_{0202} - 2\gamma_{0212} - 2\gamma_{0222}\\
&- \gamma_{1000} - \gamma_{1010} - \gamma_{1020} - \gamma_{1100} - \gamma_{1110}\\
&- \gamma_{1120} - \gamma_{1200} - \gamma_{1210} - \gamma_{1220}
\end{aligned}
$$

$$
\begin{array}{llllllllll}
+ & \gamma_{1001} & + & \gamma_{1011} & + & \gamma_{1021} & + & \gamma_{1101} & + & \gamma_{1111} \\
+ & \gamma_{1121} & + & \gamma_{1201} & + & \gamma_{1211} & + & \gamma_{1221} & & \\
- & \gamma_{2001} & - & \gamma_{2011} & - & \gamma_{2021} & - & \gamma_{2101} & - & \gamma_{2011} \\
- & \gamma_{2121} & - & \gamma_{2201} & - & \gamma_{2211} & - & \gamma_{2221} & & \\
+ & \gamma_{2002} & + & \gamma_{2012} & + & \gamma_{2022} & + & \gamma_{2102} & + & \gamma_{2112} \\
+ & \gamma_{2122} & + & \gamma_{2202} & + & \gamma_{2212} & + & \gamma_{2222} & & \\
- & \gamma_{0000} & - & \gamma_{0001} & - & \gamma_{0002} & - & \gamma_{1000} & - & \gamma_{1001} \\
- & \gamma_{1002} & - & \gamma_{2000} & - & \gamma_{2001} & - & \gamma_{2002} & & \\
+ & \gamma_{0010} & + & \gamma_{0011} & + & \gamma_{0012} & + & \gamma_{1010} & + & \gamma_{1011} \\
+ & \gamma_{1012} & + & \gamma_{2010} & + & \gamma_{2011} & + & \gamma_{2012} & & \\
- & \gamma_{0110} & - & \gamma_{0111} & - & \gamma_{0112} & - & \gamma_{1110} & - & \gamma_{1111} \\
- & \gamma_{1112} & - & \gamma_{2110} & - & \gamma_{2111} & - & \gamma_{2112} & & \\
+ & \gamma_{0120} & + & \gamma_{0121} & + & \gamma_{0122} & + & \gamma_{1120} & + & \gamma_{1121} \\
+ & \gamma_{1122} & + & \gamma_{2120} & + & \gamma_{2121} & + & \gamma_{2122} & & \\
- & \gamma_{0210} & - & \gamma_{0211} & - & \gamma_{0212} & - & \gamma_{1210} & - & \gamma_{1211} \\
- & \gamma_{1212} & - & \gamma_{2210} & - & \gamma_{2211} & - & \gamma_{2212} & & \\
- & 2\gamma_{0220} & - & 2\gamma_{0221} & - & 2\gamma_{0222} & - & 2\gamma_{1220} & - & 2\gamma_{1221} \\
- & 2\gamma_{1222} & - & 2\gamma_{2220} & - & 2\gamma_{2221} & - & 2\gamma_{2222} & & \\
- & 2\gamma_{0001} & - & 2\gamma_{0011} & - & 2\gamma_{0021} & - & 2\gamma_{1001} & - & 2\gamma_{1011} \\
- & 2\gamma_{1021} & - & 2\gamma_{2001} & - & 2\gamma_{2011} & - & 2\gamma_{2021} & & \\
- & \gamma_{0002} & - & \gamma_{0012} & - & \gamma_{0022} & - & \gamma_{1002} & - & \gamma_{1012} \\
- & \gamma_{1022} & - & \gamma_{2002} & - & \gamma_{2012} & - & \gamma_{2022} & & \\
- & \gamma_{0100} & - & \gamma_{0110} & - & \gamma_{0120} & - & \gamma_{1100} & - & \gamma_{1110} \\
- & \gamma_{1120} & - & \gamma_{2100} & - & \gamma_{2110} & - & \gamma_{2120} & & \\
- & 2\gamma_{0102} & - & 2\gamma_{0112} & - & 2\gamma_{0122} & - & 2\gamma_{1102} & - & 2\gamma_{1112} \\
- & 2\gamma_{1122} & - & 2\gamma_{2102} & - & 2\gamma_{2112} & - & 2\gamma_{2122} & & \\
- & \gamma_{0200} & - & \gamma_{0210} & - & \gamma_{0220} & - & \gamma_{1200} & - & \gamma_{1210} \\
- & \gamma_{1220} & - & \gamma_{2200} & - & \gamma_{2210} & - & \gamma_{2220} & & \\
+ & \gamma_{0202} & + & \gamma_{0212} & + & \gamma_{0222} & + & \gamma_{1202} & + & \gamma_{1212} \\
+ & \gamma_{1222} & + & \gamma_{2202} & + & \gamma_{2212} & + & \gamma_{2222}. & &
\end{array}
$$

Simplifying the above equation yields

$$
\begin{array}{rllllllllll}
\mathbb{I} = & - & 3\gamma_{0001} & - & 3\gamma_{0002} & - & 3\gamma_{0011} & - & 3\gamma_{0012} & - & 3\gamma_{0021} \\
& - & 3\gamma_{0022} & - & 3\gamma_{0102} & - & 3\gamma_{0110} & - & 3\gamma_{0111} & - & 6\gamma_{0112} \\
& - & 3\gamma_{0122} & - & 3\gamma_{0210} & - & 3\gamma_{0211} & - & 3\gamma_{0212} & - & 3\gamma_{0220} \\
& - & 3\gamma_{0221} & - & 3\gamma_{0222} & - & 3\gamma_{1000} & - & 3\gamma_{1001} & - & 3\gamma_{1002} \\
& - & 3\gamma_{1020} & - & 3\gamma_{1021} & - & 3\gamma_{1022} & - & 3\gamma_{1100} & - & 3\gamma_{1102} \\
& - & 3\gamma_{1110} & - & 3\gamma_{1112} & - & 3\gamma_{1120} & - & 3\gamma_{1122} & - & 3\gamma_{1200} \\
& - & 3\gamma_{1210} & - & 6\gamma_{1220} & - & 3\gamma_{1221} & - & 3\gamma_{1222} & - & 3\gamma_{2000} \\
& - & 6\gamma_{2001} & - & 3\gamma_{2002} & - & 3\gamma_{2011} & - & 3\gamma_{2021} & - & 3\gamma_{2100} \\
& - & 3\gamma_{2101} & - & 3\gamma_{2102} & - & 3\gamma_{2110} & - & 3\gamma_{2111} & - & 3\gamma_{2112} \\
& - & 3\gamma_{2200} & - & 3\gamma_{2201} & - & 3\gamma_{2210} & - & 3\gamma_{2211} & - & 3\gamma_{2220} \\
& - & 3\gamma_{2221}, & & & & & & & &
\end{array}
$$

which according to Eq. (5) is less than or equal to 0 and greater or equal to -6. Please note that γ's are all positive here.

APPENDIX C: THE EXACT VALUE OF EQUALITY $\mathbb{E}$

To find the exact value of the Bell expression $\mathbb{E}$, i.e. Eq. (16), let's start with

$$\mathbb{E} = +P_{11}^{00} + P_{11}^{01} + P_{11}^{10} + P_{11}^{11} + P_{12}^{22} + P_{21}^{12} - P_{22}^{12}.$$

Using appendix A to write P's in terms of γ's we get

$$\begin{aligned}\mathbb{E} = &+ \gamma_{0000} + \gamma_{0001} + \gamma_{0002} + \gamma_{0100} + \gamma_{0101}\\ &+ \gamma_{0102} + \gamma_{0200} + \gamma_{0201} + \gamma_{0202}\\ &+ \gamma_{0010} + \gamma_{0011} + \gamma_{0012} + \gamma_{0110} + \gamma_{0111}\\ &+ \gamma_{0112} + \gamma_{0210} + \gamma_{0211} + \gamma_{0212}\\ &+ \gamma_{1000} + \gamma_{1001} + \gamma_{1002} + \gamma_{1100} + \gamma_{1101}\\ &+ \gamma_{1102} + \gamma_{1200} + \gamma_{1201} + \gamma_{1202}\\ &+ \gamma_{1010} + \gamma_{1011} + \gamma_{1012} + \gamma_{1110} + \gamma_{1111}\\ &+ \gamma_{1112} + \gamma_{1210} + \gamma_{1211} + \gamma_{1212}\\ &+ \gamma_{2002} + \gamma_{2012} + \gamma_{2022} + \gamma_{2102} + \gamma_{2112}\\ &+ \gamma_{2122} + \gamma_{2202} + \gamma_{2212} + \gamma_{2222}\\ &+ \gamma_{0120} + \gamma_{0121} + \gamma_{0122} + \gamma_{1120} + \gamma_{1121}\\ &+ \gamma_{1122} + \gamma_{2120} + \gamma_{2121} + \gamma_{2122}\\ &- \gamma_{0202} - \gamma_{0212} - \gamma_{0222} - \gamma_{1202} - \gamma_{1212}\\ &- \gamma_{1222} - \gamma_{2202} - \gamma_{2212} - \gamma_{2222}.\end{aligned}$$

Simplifying the above equation would yield

$$\begin{aligned}\mathbb{E} = &+ \gamma_{0000} + \gamma_{0001} + \gamma_{0002} + \gamma_{0010} + \gamma_{0011}\\ &+ \gamma_{0012} + \gamma_{0100} + \gamma_{0101} + \gamma_{0110} + \gamma_{0111}\\ &+ \gamma_{0120} + \gamma_{0121} + \gamma_{0200} + \gamma_{0201} + \gamma_{0202}\\ &+ \gamma_{0210} + \gamma_{0211} + \gamma_{0212} + \gamma_{1000} + \gamma_{1001}\\ &+ \gamma_{1002} + \gamma_{1010} + \gamma_{1011} + \gamma_{1012} + \gamma_{1100}\\ &+ \gamma_{1101} + \gamma_{1110} + \gamma_{1111} + \gamma_{1120} + \gamma_{1121}\\ &+ \gamma_{1200} + \gamma_{1201} + \gamma_{1202} + \gamma_{1210} + \gamma_{1211}\\ &+ \gamma_{1212} + \gamma_{2002} + \gamma_{2012} + \gamma_{2022} + \gamma_{2120}\\ &+ \gamma_{2121} + \gamma_{2122} + \gamma_{2202} + \gamma_{2212} + \gamma_{2222}.\end{aligned}$$

With similar procedure, Eq. (17), i.e.

$$\mathbb{E}_c = +P_{11}^{02} + P_{11}^{12} + P_{12}^{20} + P_{12}^{21} - P_{21}^{12} + P_{22}^{12},$$

in terms of γ's would become

$$\begin{aligned}\mathbb{E}_c = &+ \gamma_{0020} + \gamma_{0021} + \gamma_{0022} + \gamma_{0120} + \gamma_{0121}\\ &+ \gamma_{0122} + \gamma_{0220} + \gamma_{0221} + \gamma_{0222}\\ &+ \gamma_{1020} + \gamma_{1021} + \gamma_{1022} + \gamma_{1120} + \gamma_{1121}\\ &+ \gamma_{1122} + \gamma_{1220} + \gamma_{1221} + \gamma_{1222}\\ &+ \gamma_{2000} + \gamma_{2010} + \gamma_{2020} + \gamma_{2100} + \gamma_{2110}\\ &+ \gamma_{2120} + \gamma_{2200} + \gamma_{2210} + \gamma_{2220}\\ &+ \gamma_{2001} + \gamma_{2011} + \gamma_{2021} + \gamma_{2101} + \gamma_{2011}\\ &+ \gamma_{2121} + \gamma_{2201} + \gamma_{2211} + \gamma_{2221}\\ &- \gamma_{0120} - \gamma_{0121} - \gamma_{0122} - \gamma_{1120} - \gamma_{1121}\\ &- \gamma_{1122} - \gamma_{2120} - \gamma_{2121} - \gamma_{2122}\\ &+ \gamma_{0202} + \gamma_{0212} + \gamma_{0222} + \gamma_{1202} + \gamma_{1212}\\ &+ \gamma_{1222} + \gamma_{2202} + \gamma_{2212} + \gamma_{2222}.\end{aligned}$$

And after simplifying we obtain

$$\begin{aligned}\mathbb{E}_c = &+ \gamma_{0020} + \gamma_{0021} + \gamma_{0022} + \gamma_{0102} + \gamma_{0112}\\ &+ \gamma_{0122} + \gamma_{0220} + \gamma_{0221} + \gamma_{0222} + \gamma_{1020}\\ &+ \gamma_{1021} + \gamma_{1022} + \gamma_{1102} + \gamma_{1112} + \gamma_{1122}\\ &+ \gamma_{1220} + \gamma_{1221} + \gamma_{1222} + \gamma_{2000} + \gamma_{2001}\\ &+ \gamma_{2010} + \gamma_{2011} + \gamma_{2020} + \gamma_{2021} + \gamma_{2100}\\ &+ \gamma_{2101} + \gamma_{2102} + \gamma_{2110} + \gamma_{2111} + \gamma_{2112}\\ &+ \gamma_{2200} + \gamma_{2201} + \gamma_{2210} + \gamma_{2211} + \gamma_{2220}\\ &+ \gamma_{2221}.\end{aligned}$$

Adding $\mathbb{E}$ and $\mathbb{E}_c$ would result

$$
\begin{aligned}
\mathbb{E}+\mathbb{E}_c = & + \gamma_{0000} + \gamma_{0001} + \gamma_{0002} + \gamma_{0010} + \gamma_{0011} \\
& + \gamma_{0012} + \gamma_{0020} + \gamma_{0021} + \gamma_{0022} + \gamma_{0100} \\
& + \gamma_{0101} + \gamma_{0102} + \gamma_{0110} + \gamma_{0111} + \gamma_{0112} \\
& + \gamma_{0120} + \gamma_{0121} + \gamma_{0122} + \gamma_{0200} + \gamma_{0201} \\
& + \gamma_{0202} + \gamma_{0210} + \gamma_{0211} + \gamma_{0212} + \gamma_{0220} \\
& + \gamma_{0221} + \gamma_{0222} + \gamma_{1000} + \gamma_{1001} + \gamma_{1002} \\
& + \gamma_{1010} + \gamma_{1011} + \gamma_{1012} + \gamma_{1020} + \gamma_{1021} \\
& + \gamma_{1022} + \gamma_{1100} + \gamma_{1101} + \gamma_{1102} + \gamma_{1110} \\
& + \gamma_{1111} + \gamma_{1112} + \gamma_{1120} + \gamma_{1121} + \gamma_{1122} \\
& + \gamma_{1200} + \gamma_{1201} + \gamma_{1202} + \gamma_{1210} + \gamma_{1211} \\
& + \gamma_{1212} + \gamma_{1220} + \gamma_{1221} + \gamma_{1222} + \gamma_{2000} \\
& + \gamma_{2001} + \gamma_{2002} + \gamma_{2010} + \gamma_{2011} + \gamma_{2012} \\
& + \gamma_{2020} + \gamma_{2021} + \gamma_{2022} + \gamma_{2100} + \gamma_{2101} \\
& + \gamma_{2102} + \gamma_{2110} + \gamma_{2111} + \gamma_{2112} + \gamma_{2120} \\
& + \gamma_{2121} + \gamma_{2122} + \gamma_{2200} + \gamma_{2201} + \gamma_{2202} \\
& + \gamma_{2210} + \gamma_{2211} + \gamma_{2212} + \gamma_{2220} + \gamma_{2221} \\
& + \gamma_{2222} \\
= & \ 1.
\end{aligned}
$$

The last equality holds due to Eq. (1). Finally as a result of positivity of γ's, we conclude that

$$|\mathbb{E}| + |\mathbb{E}_c| = 1.$$

So if according to quantum theory the value of $|\mathbb{E}_c|$ is known, then the value of $|\mathbb{E}|$ would also be known. And in the special case that $\mathbb{E}$ is positive then the exact value of $\mathbb{E}$ would be specified.